# Tutorial: Ray matrices, gaussian beams, and ABCD.app

Prof. Daniel Côté
Daniel.Cote@crulrg.ulaval.ca

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## 1 Introduction

This document is a companion guide to ABCD.app and serves as a refresher on the properties of laser beams and simple optical elements. It is not meant to be a complete discussion on ray matrices and gaussian beams since there are many excellent references on the matter and an attempt to replicate them would simply be a waste of everyone's time. Here are the important references:

1. A. E. Siegman, Lasers (Univ. Science Books, 1986), in particular Chapters 14 through 23.
2. J. W. Goodman, Introduction To Fourier Optics (Roberts & Company, 2005), Appendix B

## 2 Gaussian laser beams

Laser beams with a profile in the form of a Gaussian are important in optics because their transverse electric field profile remains Gaussian as they propagate through space. They are often simply referred to as "gaussian beams". They have interesting general properties and a formalism for easily manipulating them mathematically exist. This formalism defines a complex radius of curvature and makes use of ray matrices, sometimes called ABCD matrices. To go directly to the formalism and the mathematical definitions of the matrices, go to section 7.

## 3 Propagation of a gaussian beam

A freely propagating Gaussian laser beam, starting from a waist, will see its beam size expand according to the following equation:

$$w(z) = w_\circ \sqrt{1 + \left(\frac{z}{z_\circ}\right)^2}, \tag{1}$$

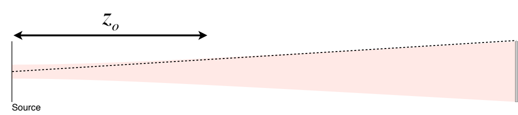


Figure 1: A laser beam, starting from its waist, will see its beam size stay approximately constant over the Rayleigh range $z_\circ$, and then expand linearly with the distance. The divergence of a beam is defined when measured at a distance larger than $z_\circ$.

where $z_\circ$ is the Rayleigh range and $z$ is the distance from the waist position, $w_\circ$ is the waist size. For calculation purposes, it is often assumed that the beam has a constant size over the Rayleigh range and a beam size that expands linearly with distance thereafter. This is shown on Fig. 1. Note that the width definition is the distance at which the electric field has reached $1/e$ of its maximum value. The width definition cannot be changed arbitrarily to the width in intensity for instance. This is important to remember since in the laboratory, the width is often measured in intensity with its full width at half maximum (FWHM).

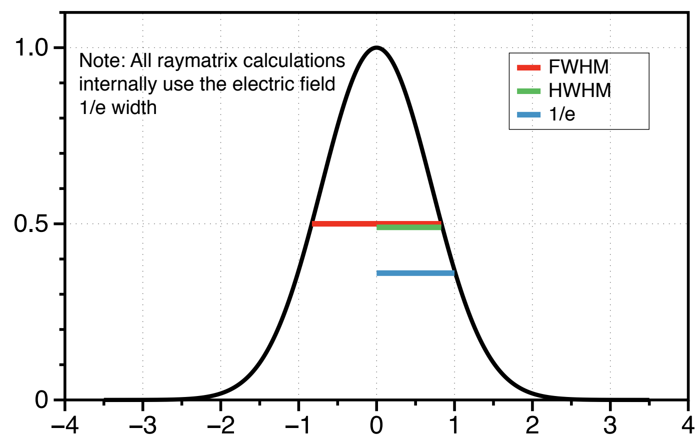


Figure 2: All calculations with ray matrices must be done with the 1/e electric field width. However, this is not necessarily the most natural measurement in the laboratory. Hence, ABCD.app offers to display the widths in various formats (FWHM, HWHM, 1/e) both in electric field or in irradiance (sometimes called intensity).

## 4 Focussing by a thin lens

When a gaussian beam passes through a lens, it will see its radius of curvature change. If the beam has a plane wavefront, it will be focussed to the focal point of the lens. On the other hand, if the

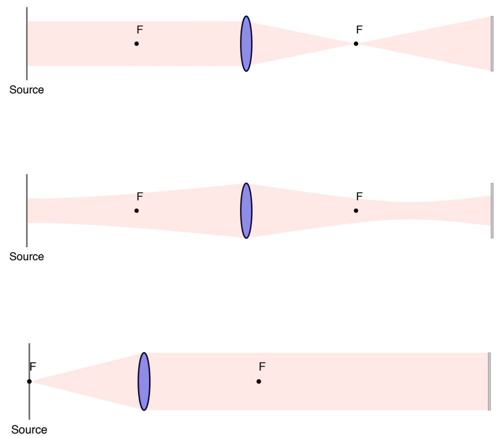


Figure 3: A plane wave will focus at the focal spot of a lens. This is not true of a beam that enters the lens with a finite radius of curvature. A point source (or very small beam) at the focal spot of a lens is collimated.

beam is not a plane wave, it will not be focussed to the focal point of the lens and will rather be focussed further away. If the beam originates from the focal point and if it is a point source (very small beam size), it will be collimated after the lens. The three cases are illustrated on Fig 3.

## 5 Focussing into a dielectric

Notice that if the material after a lens is not air and instead is a dielectric of index $n > 1$, the focal spot is not at a distance $f$ after the lens because a beam does not diffract as fast in a dielectric as it does in air. This is shown on Fig. 4.

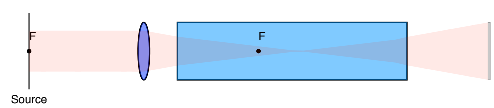


Figure 4: A lens focussing into a dielectric will produce a focal spot further away than $f$, as would be expected if the medium after the lens were air.

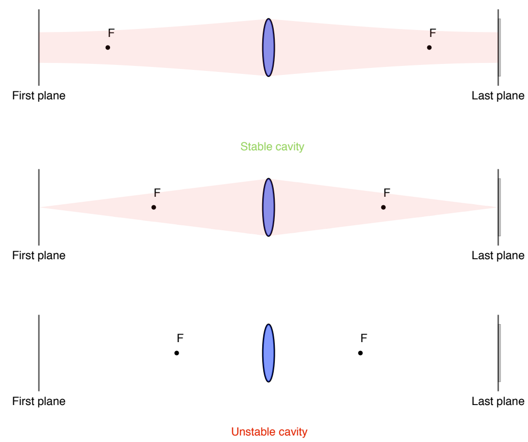


Figure 5: Laser resonator composed of a single lens and flat mirrors (not shown). Notice that the last plane is considered to be feeding directly back to the first plane. When the distance between the first and last plane is appropriate (small compared to the focal lengths of the lenses shown), the cavity is stable. When the distance is too large compared to the focal length of the lens in this example, the cavity becomes unstable, that is, it is impossible to find a beam that can go through the cavity and appear unchanged after one round trip.

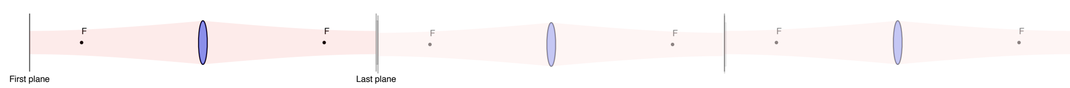


Figure 6: Note that the first plane of the cavity maps to the last plane, as if the beam path repeated itself endlessly. Also note that the beam is continuous between last and first planes.

## 6 Resonator

A resonator is a combination of optical elements through which a hypothetical beam can propagate and appear unchanged after one round trip. The beam is not user-determined: it is obtained from a calculation and is called an "eigenmode" of the cavity. It is the beam that transforms into itself after one round trip in the cavity (see Fig. 5). Currently in ABCD.app, the calculation assumes that the first plane is the start of the cavity, and the last plane maps back to the first plane, where the beam starts again (there is no assumed round trip by the program). This is illustrated on Fig. 6.

# 7 Ray matrix formalism

The general form of ray matrices is:

$$\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2}$$

and they transform complex radii of curvature $q$ :

$$q' = \frac{Aq + B}{Cq + D} \tag{3}$$

with the complex radius of curvature $q$ defined by:

$$\frac{1}{q} = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2}. \tag{4}$$

The also transform rays defined by the vector:

$$\mathbf{r} = \begin{pmatrix} r \\ \theta \end{pmatrix}, \tag{5}$$

with a standard matrix multiplication of the form:

$$\mathbf{r}' = \mathbf{Mr}, \tag{6}$$

## 7.1 Common elements

Free space of distance $d$:

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \tag{7}$$

Dielectric of length $d$ and index $n$:

$$\begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix} \tag{8}$$

Thin lens of focal length $f$:

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \tag{9}$$

Curved mirror of radius of curvature $R$ :

$$\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \tag{10}$$

Flat mirror :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{11}$$

Gaussian duct of length $L$ and $\gamma = \sqrt{n_2/n}$:

$$\begin{pmatrix} \cos\gamma L & \frac{1}{n\gamma}\sin\gamma L \\ -n\gamma\sin\gamma L & \cos\gamma L \end{pmatrix} \tag{12}$$

# 8 Definitions

## Glossary

**beam size** A transverse distance over which the electric field or the irradiance drops to a given fraction of its maximum. The beam size can be characterized according to many different definitions, but the ray matrix formalism, when used to operate on gaussian beams, must use a beam size defined as the transverse distance over which the electric field drops to $1/e$ of its maximum value.

**complex radius of curvature** The mathematical entity that includes the radius of curvature of the beam and its size, defined such that it is easily transformed by ray matrices to obtain the radius of curvature and the beam size.

**gaussian beam** A laser beam with an electric field profile shaped like a gaussian and a wavefront with a spherical radius of curvature.

**optical element** Any element that can transmit or reflect light, such as free space, a dielectric, a mirror, a thin or thick lens, a radially-varying duct.

**radius of curvature** The shape of the wavefront of a beam, or said differently, the shape relating all points of constant phase on a propagating electric field. .

**ray matrix** A $2 \times 2$ matrix that represents a simple optical elements and operates on a ray or a gaussian beam to transform it appropriately within the limits of the paraxial approximation.

**Rayleigh range** The distance over which the beam size, measured in $1/e$ in electric field, increases by a factor of $\sqrt{2}$.

**waist** Conceptually, the position at which the beam has a minimum size. When the radius of curvature and the size of any beam are known, the waist size and the waist position can be calculated. Transmission through any focussing or defocussing element will change the waist position or its size.

**waist position** The position where the waist of a beam is located. This is conceptually known as the focal spot, the focal point or the focal plane..

**waist size** The size of a gaussian beam where it is a its minimum. When the radius of curvature and the size of any beam are known, the waist size and the waist position can be calculated.

**wavefront** The surface or line that connects all points of constant phase.