Mid term 2001, notes 1

These are the notes for the mid term from 2001 (available at here , at http : // www.physics.utoronto.ca / undergraduate / PHY_ 280 F / thermal_final _ 01. pdf). I will not rewrite the question in this document, please obtain the original.

Problem 1

a) We recognize the problem of a particle in a box.nx, ny and nz must be non zero and positive. The levels and their energies are

nx	ny	nz	Energy (in units of ϵ_o)
1	1	1	3
2	1	1	6
1	2	1	6
1	1	2	6
2	2	1	9
2	1	2	9
1	2	2	9
1	1	3	11
1	3	1	11
3	1	1	11
2	2	2	12
3	2	2	17
2	3	2	17
2	2	3	17

The multiplicity is the number of states that have the same energy. It is stated in the problem that each mode (nx,ny,nz) can hold 2 fermions. The multiplicity is therefore 2, 6, 6, 6, 2 for the levels with energy 3,6,9,11 and 12.

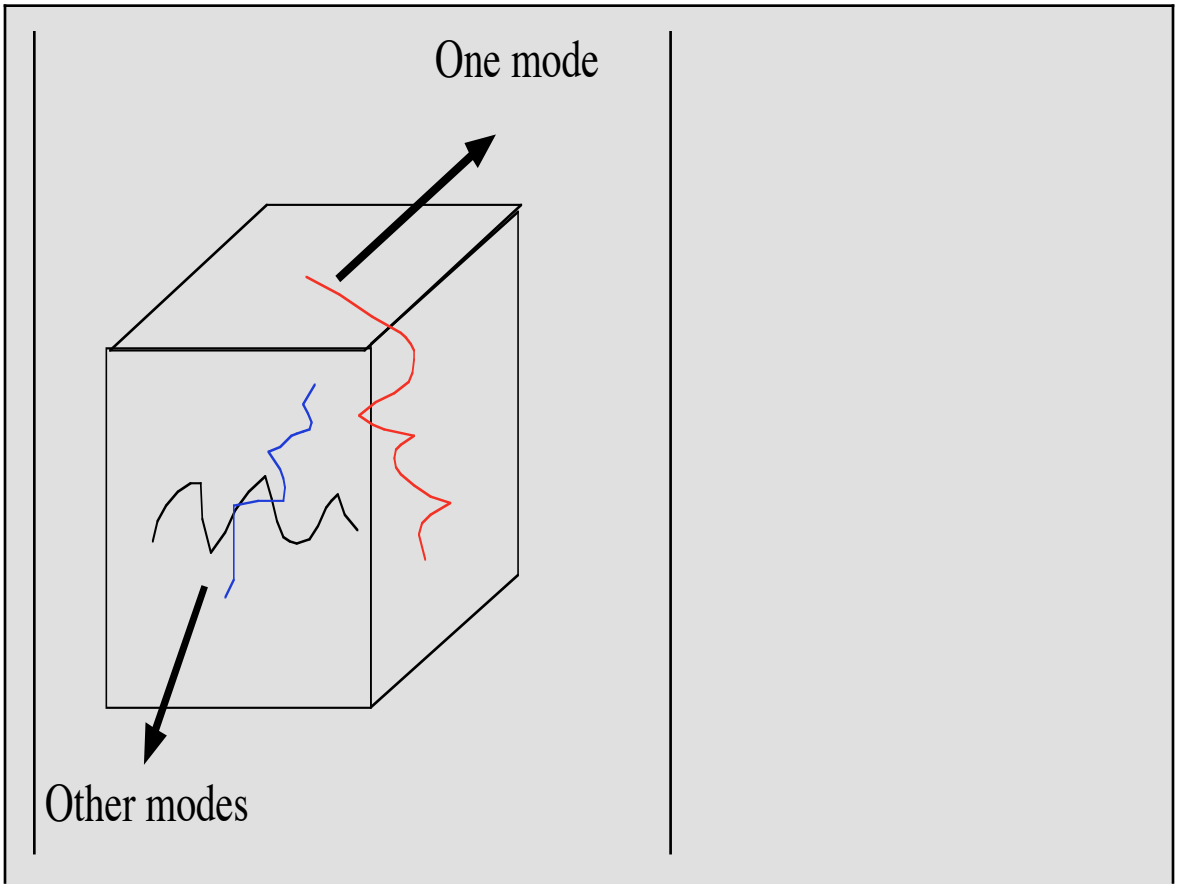
b) At zero temperature, the particles are in the lowest energy state possible. We have 10 particles,
we can only put 2 electrons in the first mode (one spin up, one spin down),
6 in the second (3 spin up, 3 spin down), and the remaining 2 in the level of energy 9 ϵ_o.

c) The Fermi energy is defined as the energy below which there are just enough orbitals to hold the number of particles assigned to the system at τ=0 K. (definition on p.183, Kittel and Kroemer). In part b) we have just written a list of all the energies at τ=0 for N=10, therefore we know the Fermi energy right away. It is the energy of the last Fermion, $\epsilon_F = 9\,\epsilon_o$. With the expression for the energy $\epsilon = \epsilon_o\,(nx^2 + ny^2 + nz^2)$ with $\epsilon_o = \frac{\hbar}{2M}\,(\frac{\pi}{L})^2$, hence the Fermi energy is:

$$\epsilon_F = \frac{9\,\hbar}{2\,M}\left(\frac{\pi}{L}\right)^2$$

Problem 5

The system we are looking at is one mode that has an energy ϵ when it is occupied. It is stated in the problem that the modes are for Fermions. Therefore we know that the mode can only accommodate 0 or 1 fermion. It is impossible to have more than 1 fermion in a given state.



a) The Gibbs partition function is defined with $\lambda = e^{\frac{\mu}{\tau}}$ and :

$$\text{zGibbs} = \sum_{\text{(all modes)}} \sum_{\text{(any \# of particles)}} \left(e^{\frac{\mu}{\tau}}\right)^{N} e^{-\frac{\epsilon_s}{\tau}}$$

There is only one mode, therefore the sum over the mode is non-existent. There are two possibilities for N and it is 0 or 1. Therefore the Gibbs partition function is given by:

$$\sum_{N=0}^{1} \left(e^{\frac{\mu}{\tau}}\right)^{N} e^{-\frac{e_s}{\tau}} = \left(e^{\frac{\mu}{\tau}}\right)^{0} e^{-\frac{0}{\tau}} + \left(e^{\frac{\mu}{\tau}}\right)^{1} e^{-\frac{\epsilon}{\tau}} = 1 + e^{\frac{\mu}{\tau}-\frac{\epsilon}{\tau}}$$

b) Looking at just one mode, we want to know the average number of particles in our mode. The average number of particle in our mode is the average number of particle in our system, because we defined the Gibbs function for a single mode. The average number of particles is the expectation value of N, that is:

$$< N > = \frac{\sum_{(\text{all modes})} \sum_{(\text{any \# of particles})} N \times \left(e^{\frac{\mu}{\tau}}\right)^N e^{-\frac{\epsilon_s}{\tau}}}{zGibbs}$$

which can be rewritten in our case as

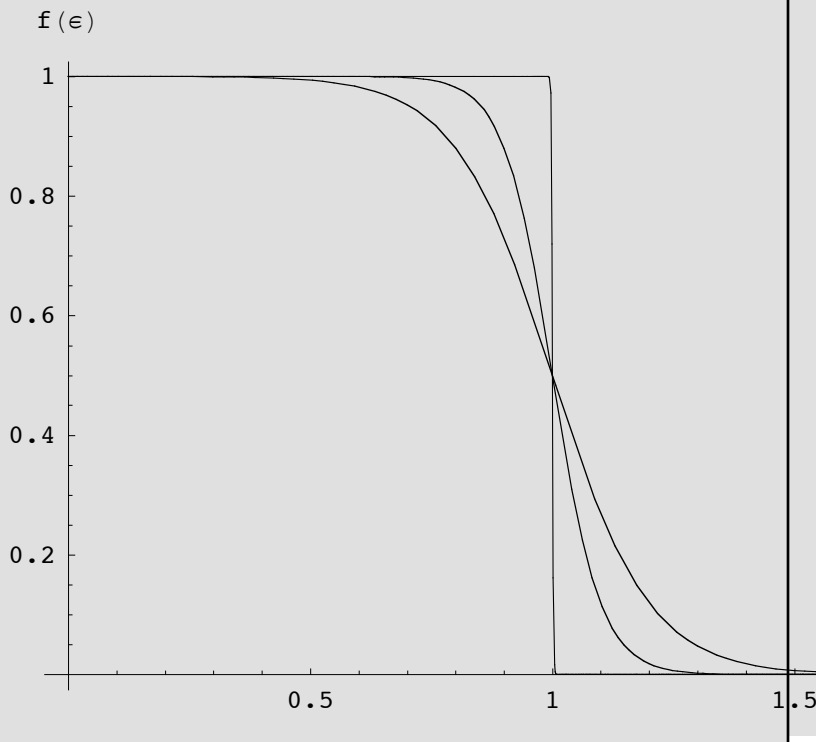
$$< N > = \frac{0 \left(e^{\frac{\mu}{\tau}}\right)^0 e^{-\frac{0}{\tau}} + 1 \left(e^{\frac{\mu}{\tau}}\right)^1 e^{-\frac{\epsilon}{\tau}}}{1 + e^{\frac{\mu}{\tau} - \frac{\epsilon}{\tau}}} = \frac{e^{\frac{\mu}{\tau}} e^{-\frac{\epsilon}{\tau}}}{1 + e^{\frac{\mu}{\tau} - \frac{\epsilon}{\tau}}} = \frac{1}{e^{\frac{\epsilon}{\tau} - \frac{\mu}{\tau}} + 1}$$

$$f[\tau\_, \epsilon\_] = \frac{1}{e^{\frac{\epsilon}{\tau} - \frac{\mu}{\tau}} + 1}$$

$$\frac{1}{1 + e^{\frac{\epsilon}{\tau} - \frac{\mu}{\tau}}}$$

The function is plotted for three temperatures and μ=1. The energy is given in units of μ. .Notice that f(ϵ) goes to 1 at ϵ=0 and that it is a step function when $\tau \to 0$.

```
Plot[{f[0.001, 1, ϵ], f[0.05, 1, ϵ], f[0.1, 1, ϵ]} ,
 {ϵ, 0, 2}, PlotRange → All, AxesLabel → {"Energy ϵ", "f(ϵ)"}]
```




```
- Graphics -
```

c) Finally, we are asked to obtain the chemical potential, knowing that

$$\text{Density}[\epsilon\_] = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}$$

$$\frac{\sqrt{2}\,V\sqrt{\epsilon}\,\left(\frac{m}{\hbar^2}\right)^{3/2}}{\pi^2}$$

The chemical potential is often obtain from the following definition:

$$\text{Ntot} = \int_0^\infty f[\tau, \epsilon]\,\text{Density}[\epsilon]\,d\epsilon$$

where f(ϵ) is the **average occupation number** of a state with an energy between ϵ and ϵ+*d*ϵ and D(ϵ) is **the density of states** or the number of states with that energy (it is equivalent to the "multiplicity" of a level in the case of continuous energy levels). The density of states is a "constant" for a system: once you have the density of states of a system, then you are done: it does not change, it does not depend on temperature. The occupation number however, does depend on

temperature, chemical potential and energy of the particle.

We are asked to calculate the chemical potential at τ=0 and we know Ntot. Therefore, we simply need to integrate over all energies and isolate μ. The chemical potential is almost always found like that.

At τ=0, f[τ,μ,ϵ]=f[0,μ,ϵ] is a step function: it is 1 if $\epsilon<\mu$, and it is 0 if the $\epsilon>\mu$ (look at the graph above: the sharp function is the step function). Hence, we can rewrite:

$$\begin{aligned}
\text{Ntot} &= \int_0^{\infty} f[\tau, \epsilon]\, \text{Density}[\epsilon]\, d\epsilon \\
&= \int_0^{\mu} f[0, \epsilon]\, \text{Density}[\epsilon]\, d\epsilon + \int_{\mu}^{\infty} f[\tau, \epsilon]\, \text{Density}[\epsilon]\, d\epsilon \\
&= \int_0^{\mu} (1) \times \text{Density}[\epsilon]\, d\epsilon \\
&= \int_0^{\mu} \text{Density}[\epsilon]\, d\epsilon
\end{aligned}$$

We obtain:

$$\text{Ntot} == \int_0^{\mu} \text{Density}[\epsilon]\, d\epsilon$$

$$\text{Ntot} == \frac{2\sqrt{2}\, V\, \mu^{3/2}\, \left(\frac{m}{\hbar^2}\right)^{3/2}}{3\pi^2}$$

and we isolate μ to get:

$$\mu == \frac{3^{2/3}\, \text{Ntot}^{2/3}\, \pi^{4/3}\, \hbar^2}{2\, m\, V^{2/3}}$$