Chapter 4 tutorial

Problem 6

The pressure is given by equation 3.26:

$$p = -\left(\frac{\partial U}{\partial V}\right)_\sigma$$

where $\left(\frac{\partial U}{\partial V}\right)_\sigma$ represents the partial derivative with respect to the volume V while keeping the entropy constant. The average energy U is :

$$U = \sum_j s_j E_j$$
$$= \sum_j s_j \hbar \omega_j$$

where s_j is the number of photons in the mode j and $E_j = \hbar \omega_j$ is the energy of the photon mode j. Note : we do not know the distribution s_j. It could be thermal equilibrium but also any other distirbution.

Note The partition function would be obtained from its definition:

$$Z = \sum_j e^{-E_j/\tau}$$

if we were at equilibrium. Because we are not at thermal equilibrium, the partition function is not useful.

By definition, we get:

$$
\begin{aligned}
p &= -\left(\frac{\partial U}{\partial V}\right)_\sigma \\
&= -\left(\frac{\partial \sum_j s_j \hbar \omega_j}{\partial V}\right)_\sigma \\
&= -\sum_j \left(\hbar \omega_j \left(\frac{\partial s_j}{\partial V}\right)_\sigma + \hbar s_j \left(\frac{\partial \omega_j}{\partial V}\right)_\sigma\right) \\
&= -\sum_j \hbar s_j \left(\frac{\partial \omega_j}{\partial V}\right)_\sigma
\end{aligned}
$$

$\hbar \omega_j \left(\frac{\partial s_j}{\partial V}\right)_\sigma$ is zero, but I am not entirely sure why unless we are at equilibrium (and therefore s_j does not change). In general, I am not sure why it would be zero.

We know from Eq. 4.15 that:

$$
\omega_j = \frac{j \pi c}{L}
$$

therefore, using chain rules, we can calculate:

$$
\begin{aligned}
\left(\frac{\partial \omega_j}{\partial V}\right)_\sigma &= \frac{\partial \omega_j}{\partial L} \frac{\partial L}{\partial V} \\
&= \frac{\partial \omega_j}{\partial L} \Big/ \frac{\partial V}{\partial L} \\
&= -\frac{j \pi c}{L^2} \Big/ \frac{\partial (L^3)}{\partial L} \\
&= -\frac{j \pi c}{L^2} \Big/ 3 L^2 \\
&= -\omega_j / 3 L^3 \\
&= -\omega_j / 3 V
\end{aligned}
$$

Note: if the volume is increased, the energy (or frequency) of a mode decreases as we can see from the sign of the derivative.

We substitute and obtain:

$$
p = \sum_j \hbar s_j \omega_j / 3 V = U / 3 V
$$

Now, we are asked to compare the kinetic pressure with the thermal radiation pressure. Because we now deal with thermal radiation pressure,(i. e. pressure of radiation at thermal equilibirum), we can use the results of p. 94 in the page, which are valid for thermal equilibrium. We know that at thermal equilibrium, the energy density of radiation is:

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ThermalRadiationEnergyDensityUoverV[T_] = π^2/(15 ℏ^3 c^3) (k T)^4
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$$\frac{3.05009\times10^{-18}\ \text{Joule}\ T^4}{\text{Kelvin}^4\ \text{Meter}^3}$$

And therefore, using the derived result p=U/3V, we can say that the thermal radiation pressure is:

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p_tr = ThermalRadiationEnergyDensityUoverV[2 × 10^7 Kelvin] / 3
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$$\frac{1.62672\times10^{11}\ \text{Joule}}{\text{Meter}^3}$$

The pressure of an ideal gas at that temperature is given by Eq. (4.74):

$$p_{ig} = \left(\frac{N}{V}\right) k\,T = C\,k\,T$$

where C is the concentration of molecules per volume. With C= $\frac{1\ \text{Mole}}{(0.01\ \text{Meter})^3}$, the pressures are equal when:

$$C\,k\,T == \frac{1}{3}\,\frac{\pi^2}{15\,\hbar^3\,c^3}\,(k\,T)^4$$

and therefore

$$T == \sqrt[3]{\frac{45\,\hbar^3\,c^3}{\pi^2\,k^3}\,\frac{1\ \text{Mole}}{(0.01\ \text{Meter})^3}}$$

$$T == 2.01472\times10^8\ (\text{Kelvin}^3)^{1/3}$$

In the sun, the kinetic pressure is roughly:

$$p_{ig} == \frac{1\ \text{Mole}}{(0.01\ \text{Meter})^3}\ 2\times 10^7\ \text{Kelvin}\times k$$

$$p_{ig} == \frac{1.66289\times 10^{14}\ \text{Joule}}{\text{Meter}^3}$$

and in the center it is a hundred times more:

$$p_{ig} == \frac{100\ \text{Mole}}{(0.01\ \text{Meter})^3}\ 2\times 10^7\ \text{Kelvin}\times k$$

$$p_{ig} == \frac{1.66289\times 10^{16}\ \text{Joule}}{\text{Meter}^3}$$

Note : A Joule is a Newton × Meter. We therefore have the units of Newton × Meter / Meter^3 = Newton / Meter^2 which is a Pascal.

Problem 7

We rewrite the sum as an integral in spherical coordinates. A sum over the states labelled by $\vec{n} = (n_x, n_y, n_z)$. The sum over all elements is the same as as sum over the elementary volume element $dn_x\, dn_y\, dn_z$. However , it is not convenient to use Cartesian coordinates because the energy is given as a function of n, not (n_x, n_y, n_z). Therefore we go to Soehrical coordinates where the elementary volume is element $n^2\ \text{Sin}[\theta]\ dn\, d\theta\, d\phi$. The integrals over $\text{Sin}[\theta]\ d\theta\, d\phi$ are trivial. Finally, we use the following Taylor series :

$$\text{Log}\left[1 - e^{-\frac{k n \hbar \varpi}{\tau}}\right] = -\sum_{k=1}^{\infty} \frac{e^{-\frac{k n \hbar \varpi}{\tau}}}{k}$$

Finally, we note that there are two polarizations per mode, so we must multiply by 2 to obtain the number of states.

$$= 2\times\tau \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty} \text{Log}\left[1 - e^{-\frac{k n \hbar \varpi}{\tau}}\right]\ n^2\ \text{Sin}[\theta]\ dn\, d\theta\, d\phi$$

$$= \pi\,\tau \int_0^{\infty} n^2 \left(-\sum_{k=1}^{\infty} \frac{e^{-\frac{k n \hbar \varpi}{\tau}}}{k}\right) dn$$

We substitute for clarity:

$$x \equiv \frac{n \hbar \varpi}{\tau}$$
$$dx = \frac{\hbar \varpi}{\tau} dn$$

to obtain:

$$= -\frac{\pi \tau^4}{\hbar^3 \varpi^3} \int_0^\infty x^2 \sum_{k=1}^{\infty} \frac{e^{-k x}}{k} \, dx$$

We integrate by parts with $\int u \, dv = uv - \int v \, du$, and we use $u = x^2$,

$dv = \sum_{k=1}^{\infty} \frac{e^{-k x}}{k} \, dx$, therefore we have $du = 2 x \, dx$ and $v = -\sum_{k=1}^{\infty} \frac{e^{-k x}}{k^2}$:

$$= -\frac{\pi \tau^4}{\hbar^3 \varpi^3} \left(\left(x^2 (-1) \sum_{k=1}^{\infty} \frac{e^{-k x}}{k^2} \right)_{x=0}^{x=\infty} - \int_0^\infty 2 x (-1) \sum_{k=1}^{\infty} \frac{e^{-k x}}{k^2} \, dx \right)$$
$$= -\frac{\pi \tau^4}{\hbar^3 \varpi^3} \left(\int_0^\infty 2 x \sum_{k=1}^{\infty} \frac{e^{-k x}}{k^2} \, dx \right)$$

We integrate by parts with $u = 2x$, $dv = \sum_{k=1}^{\infty} \frac{e^{-k x}}{k^2} \, dx$, therefore we have $du = 2 \, dx$ and $v = -\sum_{k=1}^{\infty} \frac{e^{-k x}}{k^3}$:

$$= -\frac{\pi \tau^4}{\hbar^3 \varpi^3} \left(\left(2 x \sum_{k=1}^{\infty} \frac{e^{-k x}}{k^3} \right)_{x=0}^{x=\infty} - \int_0^\infty 2 (-1) \sum_{k=1}^{\infty} \frac{e^{-k x}}{k^3} \, dx \right)$$
$$= -\frac{2 \pi \tau^4}{\hbar^3 \varpi^3} \int_0^\infty \sum_{k=1}^{\infty} \frac{e^{-k x}}{k^3} \, dx$$
$$= -\frac{2 \pi \tau^4}{\hbar^3 \varpi^3} (-1) \left(\sum_{k=1}^{\infty} \frac{0}{k^4} - \sum_{k=1}^{\infty} \frac{1}{k^4} \right)$$

We can actually find the limit of the series:

$$\sum_{k=1}^{\infty} \frac{1}{k^4}$$

$$\frac{\pi^4}{90}$$

Therefore we obtain the final result:

$$F = -\frac{2\,\pi\,\tau^4}{\hbar^3\,\varpi^3}\;\frac{\pi^4}{90}$$

$$-\frac{\pi^5\,\tau^4}{45\,\varpi^3\,\hbar^3}$$